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OPTIMUM PORTFOLIO DIVERSIFICATION IN A GENERAL CONTINUOUS-TIME MODEL

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The problem of determining optimal portfolio rules is considered. Prices are allowed to be stochastic processes of a fairly general nature, expressible as stochastic integrals with respect to semimartingales. The set of stochastic differential equations assumed to describe the price behaviour still allows us to handle both the associated control problems and those of statistical inference.

The greater generality this approach offers compared to earlier treatments allows for a more realistic fit to real price data. with the obvious implications this has for the applicability of the theory.

The additional problem of including consumption is also considered in some generality. The associated Bellman equation has been solved in certain particular situations for illustration. Problems with possible reserve funds, borrowing and shortselling might be handled in the present framework.

The problem of statistical inference concerning the parameters in the semimartingale price processes will be treated elsewhere.

portfolio selection * stochastic control * martingales * stochastic integrals

1. Introduction

In this paper we consider optimum portfolio and consumption rules in a stochastic framework. We allow a fairly general model for the prices on assets, namely that the prices are semimartingales consisting of both continuous components and discrete jumps. In the time continuous case the geometric Brownian motion process has been a common model for prices in perfect markets. Also, more general, strong Markov continuous diffusion processes have been considered, like the Ornstein–Uhlenbeck process (see e.g. Merton [25]). One advantage with diffusion models is the few parameters that have to be estimated from the data. Practical situations can easily occur where those processes are not flexible enough to model the actual price behaviour. This will particularly be the case if one observes major price changes. Hence it is desirable to allow for jumps to occur at random times in addition to the continuous component in the price processes. Inclusion of a point process

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component into the price model thus makes the model more realistic, and still the number of parameters is small enough for the estimation procedure to make sense. Now we have to estimate the associated intensities of the imbedded point processes in addition to the drift and diffusion parameters of the continuous component.

Thus we justify the price models from a time-series point of view, i.e. we want a 'good fit' to real data. Another reason for this price structure is that the return on a portfolio can be very conveniently represented as a stochastic integral with respect to the prices. These matters are discussed in a general setting by Harrison and Pliska [43]. Among other things they show that the security market is complete if and only if its vector price process has a certain martingale representation property. However, we shall not discuss contingent claim valuation and complete markets.

Despite the fact that the price model is now of a very general nature, we are still able to handle the 'control problem', i.e. determine the optimal portfolio strategy. Basically what we then need is a generalized Ito's lemma to handle the 'chain rule' when we introduce the utility function into the optimization procedure. It turns out that the Markov property for prices is no longer essential for the control procedure to work.

One utility function used in the portfolio problem will be given special attention, namely the natural logarithm of the wealth (the Kelly criterion). In the present continuous parameter problem it can be shown to have certain optimal properties, considered as a normative utility function (see Thorp [37]). Thus some discrete time results (see Breiman [8]) can be extended.

The treatment in Section 2 will be used in Section 3, where we focus on the optimization problem. In the portfolio problem it turns out that the computational aspects of the Kelly criterion prove preferable to work with, since we then avoid the problem of solving the Bellman equation (see Section 3).

In Section 4 we also include the problem of optimal consumption in addition to the portfolio diversification problem. Certain special cases are worked out for illustration. Problems with a 'gambler's reserve fund', borrowing and shortselling are discussed as well.

The classical referernces to this kind of analyses are Samuelson [34], Merton [25, 26], Mossin [28], Tobin [40], Markowitz [24], Hakanson [18], among others. For a more modern treatment along the lines of this paper, we refer again to Harrison and Pliska [43].

The Kelly criterion includes that the relative risk aversion is constant (iso-elastic marginal utility), which implies that one's attitude towards financial risk is independent of one's wealth level [34, 25]. It is to be stressed that the present treatment is not confined only to logarithmic utility, but outlines the general theory for any twice differentiable utility function. However, only a few additional cases are solved in detail to indicate what can be anticipated. In this kind of analysis, once the complicated stochastic optimization problem is reduced to a problem in ordinary analysis, it is considered solved (even if the resulting differential equation can not be solved in general with the present techniques).

2. The model and the problem

Suppose an agent is faced with n different investment alternatives. The prices $p_i(t)$ at time t, $i=1,2,\ldots,n$ of the different assets are stochastic processes with sample paths being continuous from the right, having left hand limits (CADLAG). In addition we assume that the price processes permit the following type of stochastic differential equations

$$\frac{\mathrm{d}p_i(t)}{p_i(t-)}=\mu_i(t)\,\mathrm{d}t+\sigma_i(t)\,\mathrm{d}b_i(t)+\sum_{k=-m}^{m}\beta_{ik}\,\mathrm{d}N_{ik}(t), \quad i=1,2,\ldots,n. \tag{1}$$

Denote by $\boldsymbol{p}(t)=(p_1(t),p_2(t),\ldots,p_n(t))$, let (Ω,F,P) be a probability space and F_t a filtration of σ-algebras, $F_t\subset F$. Here $b(t)=(b_1(t),b_2(t),\ldots,b_n(t))$ is a Wiener process and $N_{ij}(t)$ are point processes, adapted to $\{F_t\}$. $\mu_i(t)$, $\sigma_i(t)$ are measurable functionals, possibly depending on the past of the process $\boldsymbol{p}(s)$, $s\leq t$ as well as on time t itself (i.e. nonanticipative). Here μ_i is the instantaneous conditional expected change in price p_i per unit time as a fraction of the current price for the continuously varying part of the price p_i, σ_i^2 is the instantaneous conditional variance per unit time as a fraction of p_i^2 for the continuous sample part of p_i. In addition, we allow the prices to jump at random times. Here $N_{ij}(t)$ is the number of price changes of size β_{ij}, expressed as a fraction of the current price, that occur during $[0,t]$ for asset i, $j=-m,-m+1,\ldots,m-1,m$, where we assume

$$\beta_{i,(-m)}<\beta_{i,(-m+1)}<\cdots<0\leq\beta_{i0}<\beta_{i1}<\cdots<\beta_{im}$$

$i=1,2,\ldots,n$. The β_{ij}'s are given numbers. The point process component of the price is often referred to as a marked point process (see Brillinger [10]).

The formalism that (1) represents is in the framework of stochastic integration of Ito type (see McKean [23], Meyer [27], Gihman and Skorohod [16, 17] etc.).

We will in addition assume that there exists intensity processes $\lambda_{ij}(t)$ such that

$$M_{ij}(t)=N_{ij}(t)-\int_0^t\lambda_{ij}(s)\,\mathrm{d}s \tag{2}$$

are martingales (see e.g. Meyer [27]). These intensities have the following interpretation:

$$\lambda_{ij}(t)\,\mathrm{d}t=P[\mathrm{d}N_{ij}(t)=1\mid F_t] \tag{3}$$

(see e.g. Brillinger [10]). These intensity processes will be considered as unknown and will have to be estimated from the price data together with the μ_i's and the σ_i's.

Conditions necessary for (1) to possess a solution are the following: The functions μ_i, σ_i and λ_{ij} are nonanticipative, adapted to $\{F_t, t\in[0,T]\}$,

$$\sum_{i=1}^{n}\int_0^T\sigma_i(t)^2\,\mathrm{d}t<\infty, \qquad \sum_{i=1}^{n}\sum_{k=-m}^{m}\int_0^T\beta_{ik}^2\lambda_{ik}(t)\,\mathrm{d}t<\infty$$

and they must be linearly bounded, i.e.

$$\sum_{i=1}^{n} \mu_i(t)^2 + \sum_{i=1}^{n} \sigma_i(t)^2 + \sum_{i=1}^{n} \sum_{k=-m}^{m} \beta_{ik}^2 \lambda_{ik}(t) \leq c(1 + \|p(\cdot)\|_t^2)$$

where c is a constant and $\|p(\cdot)\|_t = \sup_{0 \leq s \leq t} |p(s)|$, $t \in [0, T]$. (See Gihman and Skorohod [16].)

The correlation structure between the prices in the present framework is contained in the terms μ, σ and λ which all can depend on the joint price history $p(s)$, $s \leq t$. This way we need no instantaneous correlation matrix between the Wiener processes b_i (see Merton [26]). Technically this matrix plays no role.

Now, denote the wealth of the agent at time t by W_t. The change in wealth at time t satisfies

$$\mathrm{d}W_t = W_t \sum_{i=1}^{n} \rho_i(t, \boldsymbol{p}) \frac{\mathrm{d}p_i(t)}{p_i(t-)} \tag{4}$$

where $\rho_i(t, \boldsymbol{p})$ is the fraction of the agent's wealth that is invested in alternative i at time t. As the notation indicates, this is a functional of the past of the price process, measurable with respect to F_t, and satisfies

$$\sum_{i=1}^{n} \rho_i(t, \boldsymbol{p}) = 1 \quad \text{for all } t. \tag{5}$$

If shortselling and borrowing is not allowed, we need in addition to (5) also

$$\rho_i(t, \boldsymbol{p}) \geq 0, \quad i = 1, 2, \ldots, n, \ t \geq 0. \tag{6}$$

For a discussion of why (4) is the 'correct' way to define the return process in continuous time, we refer to Merton [25] for the continuous part of the associated stochastic integral and to Harrison and Pliska [43] for the combined stochastic integral we are dealing with.

Given a time horizon $T < \infty$ and a concave function $U : \mathbb{R} \to \mathbb{R}$, the problem is to choose a strategy $\boldsymbol{\rho}^*$ so as to maximize $E[U(W_T)]$. A special case, called the Kelly criterion, is when $U(W) = \log W$. From discrete time portfolio theory it is known that this utility function has certain optimal properties when it comes to achieving certain stated goals (see Breiman [9]). These properties can be shown to carry over to continuous time as well.

The mean-variance analysis of Tobin [40, 39] and Markowitz [24] will in fact be used if we try to solve the quadratic case, i.e. if we use only the two first terms in the series expansion of the logarithm function. In that case we solve a trade-off problem, namely to maximize expected wealth and minimize a certain function of the risk. Samuelson [33, 35] has emphasized the lack of generality of mean-variance analyses. He suggested that most of the interesting propositions of risk theory can be proved for the general case with no approximations being involved. This trend will be followed in the present paper where we solve the exact problem. However, in continuous time these problems partly disappear, as will be demonstrated later.

To see heuristically why the Kelly criterion is crucial let $U(t) = \log W(t)$. Formally we can define the following:

$$\text{Rate of growth} = \exp\left[\frac{1}{t}\int_0^t \mathrm{d}U(s)\right] - 1,$$

$$\text{Average return} = \frac{1}{t}\int_0^t W_{s-}^{-1}\,\mathrm{d}W(s).$$

(The formal interpretation of these integrals is given in Section 3.) By maximizing $E\{\int_0^t \mathrm{d}U(s)/t\}$, we are also certain to make the expected rate of growth large as well since by Jensen's inequality

$$E\left\{\exp\left[\frac{1}{t}\int_0^t \mathrm{d}U(s)\right] - 1\right\} \geq \exp\left\{E\left[\frac{1}{t}\int_0^t \mathrm{d}U(s)\right]\right\} - 1.$$

However, by maximizing the expected average return it does not follow that the expected rate of growth is large. Also, by maximizing the expected rate of growth, the expected average return becomes large. These two claims may again be shown by Jensen's inequality by going back to the definitions of stochastic integrals (the technical details are omitted).

Let us now proceed directly, and find an equation for $E[U(W_T)]$:

The price on asset i can be written as follows:

$$p_i(t) = p_i(0) + \int_0^t p_i(s-)\sigma_i\,\mathrm{d}b_i(s) + \int_0^t p_i(s-)\sum_{k=-m}^{m} \beta_{i,k}\,\mathrm{d}M_{ik}(s)$$
$$+ \int_0^t \left\{p_i\left(\mu_i + \sum_{k=-m}^{m} \beta_{ik}\lambda_{ik}\right)\right\}\mathrm{d}s. \tag{7}$$

Here we notice that

$$p_i(t) = p_i(0) + p_i^M(t) + p_i^A(t)$$

with

$$p_i^M(t) = p_i^{Mc}(t) + p_i^{Md}(t)$$

where

$$p_i^{Mc}(t) = \int_0^t p_i(s-)\sigma_i\,\mathrm{d}b_i(s)$$

and

$$p_i^{Md}(t) = \int_0^t p_i(s-)\sum_{k=-m}^{m} \beta_{ik}\,\mathrm{d}M_{ik}(s)$$

are two martingales, the first one with continuous sample paths, the second one containing jumps. Further

$$p_i^A(t) = \int_0^t \left\{(p_i\mu_i) + \sum_{k=-m}^{m} p_i\beta_{ik}\lambda_{ik}\right\}^+ \mathrm{d}s - \int_0^t \left\{(p_i\mu_i) + \sum_{k=-m}^{m} p_i\beta_{ik}\lambda_{ik}\right\}^- \mathrm{d}s$$

where the two terms are both increasing processes.

For the process W we see that

$$W(t) = W(0) + \int_0^t \sum_{i=1}^n W_{s-}\rho_i \left(\sigma_i \, \mathrm{d}b_i + \sum_{k=-m}^m \beta_{ik} \, \mathrm{d}M_{ik} \right)$$
$$+ \int_0^t \sum_{i=1}^n W\rho_i \left(\mu_i + \sum_{k=-m}^m \beta_{ik}\lambda_{ik} \right) \mathrm{d}s. \tag{8}$$

Here

$$W(t) = W(0) + W^M(t) + W^A(t)$$

with

$$W^M(t) = W^{Mc}(t) + W^{Md}(t)$$

and

$$W^{Mc}(t) = \int_0^t \sum_{i=1}^n W_{s-}\rho_i\sigma_i \, \mathrm{d}b_i(s),$$

$$W^{Md}(t) = \int_0^t \sum_{i=1}^n \sum_{j=-m}^m \beta_{i,j} W_{s-}\rho_i \, \mathrm{d}M_{ij}(s)$$

are the two martingales, the first with continuous paths, the second of discontinuous type. Also $W^A(t)$ is the difference of two increasing predictable processes.

We now need the generalized Ito's lemma. Let X be a semimartingale, and denote its continuous part by X^c. Then we use the notation

$$\langle X^c, X^c \rangle_t = \lim_n \sum_j E\{(X^c_{t^n_{j+1}} - X^c_{t^n_j})^2 \mid F_{t^n_j}\}$$

where the lim signifies that the partition mesh of $[0, t]$ goes to zero, and the convergence takes place in L^1.

From Ito's lemma and the price structure given in (1) (or (7)), we want to derive the expression to be maximized. Suppose U is any utility function satisfying the requirements of Ito's lemma. From (8) we get

$$\langle W^{Mc}, W^{Mc} \rangle_t = \int_0^t \sum_{i=1}^n (W\rho_i\sigma_i)^2 \, \mathrm{d}s$$

and Ito's generalized lemma yields

$$U(W_T) = U(W_0) + \int_0^T \sum_{i=1}^n \frac{\partial U}{\partial W}(W_{t-})[W_t\rho_i\mu_i \, \mathrm{d}t + W_{t-}\rho_i \, \mathrm{d}b_i]$$
$$+ \int_0^T \sum_{i=1}^n \frac{\partial U}{\partial W}(W_{t-})W_{t-}\rho_i \sum_{k=-m}^m \beta_{ik} \, \mathrm{d}N_{ik} - \sum_{t \le T} \frac{\partial U}{\partial W}(W_{t-})(W_t - W_{t-})$$
$$+ \tfrac{1}{2} \int_0^T \frac{\partial^2 U}{\partial W^2}(W_{t-}) \sum_{i=1}^n (W\rho_i\sigma_i)^2 \, \mathrm{d}t + \sum_{t \le T} (U(W_t) - U(W_{t-})).$$

The third and fourth term on the right-hand side cancel, since at a time t of jump of $N_{ij}(t)$, $t=\tau_{ij}$ ($\tau_{ij}=\infty$ if no such jump takes place), $W_t - W_{t-} = W_{t-}\rho_i\beta_{ij}$. Hence the fourth term can be written

$$\sum_{i=1}^{n}\sum_{j=-m}^{m}\sum_{\tau_{ij}\le T}\frac{\partial U(W_{\tau_{ij}-})}{\partial W}(W_{\tau_{ij}}-W_{\tau_{ij}-})$$
$$=\int_0^T\sum_{i=1}^{n}\sum_{j=-m}^{m}\frac{\partial U(W_{t-})}{\partial W}W_{t-}\rho_i\beta_{ij}\,\mathrm{d}N_{ij}(t)$$

which equals the third term.

The last term can be written

$$\sum_{i=1}^{n}\sum_{j=-m}^{m}\left\{\sum_{\tau_{ij}\le T}U(W_{\tau_{ij}-}+W_{\tau_{ij}}\beta_{ij}\rho_i)-U(W_{\tau_{ij}-})\right\}$$
$$=\int_0^T\sum_{j=1}^{n}\sum_{j=-m}^{m}(U(W_{t-}+W_t\beta_{ij}\rho_i)-U(W_{t-}))\,\mathrm{d}N_{ij}(t)$$
$$=\int_0^T\sum_{j=1}^{n}\sum_{j=-m}^{m}\{U(W_{t-}+W_t\beta_{ij}\rho_i)-U(W_{t-})\}(\mathrm{d}M_{ij}(t)+\lambda_{ij}(t)\,\mathrm{d}t).$$

Upon taking expectations, the martingales disappear and we are left with

$$EU(W_T)=U(W_0)+E\int_0^T\sum_{j=1}^{n}\left\{\frac{\partial U}{\partial W}(W_t)(W\rho_i\mu_i)_t+\frac{1}{2}\frac{\partial^2 U}{\partial W^2}(W_t)(W\rho_i\sigma_i)_t^2\right.$$
$$\left.+\sum_{j=-m}^{m}(U(W_t+W_t\beta_{ij}\rho_i)-U(W_t))\lambda_{ij}(t)\right\}\mathrm{d}t. \tag{9}$$

Here we also substituted W_t for W_{t-} throughout since the set of time points where the process W jumps in $[0, T]$ has Lebesque measure zero. Hence our optimization problem consists in determining the policy $\boldsymbol{\rho}^*$ such that the above integrand takes on its constrained maximum for each t subject to (5) (or to (5) and (6)).

In the case of the Kelly criterion we get:

$$\frac{\partial^2}{\partial W^2}\log(W_t)=-\frac{1}{W_t^2},$$

and

$$\frac{1}{2}\int_0^T\frac{\partial^2 U}{\partial W^2}(W_t)\sum_{i=1}^{n}(W_t\rho_i\sigma_i)^2\,\mathrm{d}t=\frac{1}{2}\int_0^T\sum_{i=1}^{n}\rho_i^2\sigma_i^2\,\mathrm{d}t,$$

while $(\partial/\partial W)\log(W_t)=1/W_t$, which leaves the first term in (9) as $\int_0^T\sum_{j=1}^{n}\rho_i\mu_i\,\mathrm{d}t$. Furthermore we see that

$$\int_0^T\sum_{i=1}^{n}\sum_{j=-m}^{m}(\log(W_t+W_t\beta_{ij}\rho_i)-\log(W_t))\lambda_{ij}(t)\,\mathrm{d}t$$
$$=\int_0^T\sum_{i=1}^{n}\sum_{j=-m}^{m}(\log(1+\beta_{ij}\rho_i))\lambda_{ij}(t)\,\mathrm{d}t$$

which leaves us with

$$E[\log(W_T)] = \log(W_0) + \sum_{i=1}^{n} \int_0^T \left(\rho_i \mu_i - \tfrac{1}{2}\sigma_i^2 \rho_i^2 + \sum_{j=-m}^{m} (\log(1 + \rho_i \beta_{ij}))\lambda_{ij} \right) \mathrm{d}t. \tag{10}$$

In this case we could in fact have derived this expression directly by using Taylor series expansion of the log function and some intuition when it comes to 'multiplying' differentials. The techniques of nonstandard analyses could probably be used in a formal proof of (10) as well (see e.g. Lindstrøm [22]).

One reason for incorporating the jump processes into the price model is that we want to be able to model more drastic changes in the price behaviour than the Ito-processes can account for alone. The common geometric Brownian motion process is obtained for asset i by setting $\beta_{ik} = 0$, $k = -m, \ldots, m$, $\mu_i = c_{i1}$, $\sigma_i = c_{i2}$ where c_{i1} and c_{i2} are two constants. Likewise, the Ornstein–Uhlenbeck process is obtained for asset i by setting $\beta_{ik} = 0$, $k = -m, \ldots, m$, $\mu_i = c_{i1}$ and $\sigma_i = c_{i2}/p_i$. Both these are strong Markov, diffusion type processes. We will see later that in the present setting we do not have to limit ourselves to Markov type price behaviour. On the other hand a pure point process will result if we take $\mu_i = 0$ and $\sigma_i = 0$. This does not mean that we have a risk free asset. In the present setting the risk is $\tilde{\sigma}_i$ given by

$$\tilde{\sigma}_i^2(t, \boldsymbol{p}) = \sigma_i^2(t, \boldsymbol{p}) + \sum_{k=-m}^{m} \beta_{ik}^2 \lambda_{ik}(t, \boldsymbol{p}) \tag{11}$$

which can be found by direct computation, using the martingale properties of b_i and M_{ij}, noting that

$$E\left[\int_0^t \beta_{ij} dM_{ij}\right]^2 = E\left[\beta_{ij}^2 \int_0^t \lambda_{ij}(u, \boldsymbol{p})\, \mathrm{d}u\right]$$

and

$$E\left[\int_0^t \sigma_i(s, \boldsymbol{p}) db_i\right]^2 = E\left[\int_0^t \sigma_i^2(s, \boldsymbol{p})\, \mathrm{d}s\right]$$

(see e.g. Meyer [27], McKean [23]).

3. The optimization problem and examples

We now return to the general problem of finding an optimal portfolio strategy ρ^* for any concave utility function leading to (9). First we observe that this problem is more complex than in the case of logarithmic utility. From (9) it follows that the integrand does depend on the wealth W_t itself in the general case. Here we need the principles of Bellman's dynamic programming to attack our problem:

$$\sup_{\rho \in U_{[0,T]}} E \int_0^T \sum_{i=1}^n \left\{ \frac{\partial U}{\partial W}(W_t)(W_t \rho_i \mu_i) + \tfrac{1}{2} \frac{\partial^2 U}{\partial W^2}(W_t)(W_t \rho_i \sigma_i)^2 \right.$$
$$\left. + \sum_{j=-m}^{m} [U(W_t(1+\beta_{ij}\rho_i)) - U(W_t)]\lambda_{ij} \right\} \mathrm{d}t. \tag{12}$$

Here $U_{[0,T]}$ is the set of permissible portfolio rules on $[0, T]$. We now want to make use of control theory, and for that to work in practice we often need our processes to be Markovian. Hence, in the present setting we assume the following to hold

$$\mu_i(t) = \mu_i(t, \boldsymbol{p}(t)), \quad \sigma_i(t) = \sigma_i(t, \boldsymbol{p}(t)),$$
$$\lambda_{ij}(t) = \lambda_{ij}(t, \boldsymbol{p}(t)), \qquad i = 1, 2, \ldots, n, \ j = -m, -m+1, \ldots, m.$$

Let us now define

$$Z(t, W) = \sup_{\rho \in U_{[t,T]}} E[U(W) \mid W_t = W, F_t] \tag{13}$$

Here $U_{[t,T]}$ are the permissible portfolio rules on the time interval $[t, T]$. (The technical concept hidden in the term 'permissible' is basically (5) and a requirement that $\boldsymbol{\rho}$ is nonanticipative; see Gihman and Skorohod [16]).

Then, the Bellman equation associated to the problem in (12) is

$$\sup_{\rho \in U_{[t,T]}} \left\{ \frac{\partial Z(t, W)}{\partial W} \sum_{i=1}^n W\rho_i(t, \boldsymbol{p})\mu_i(t, \boldsymbol{p}) + \frac{1}{2} \frac{\partial^2 Z(t, W)}{\partial W^2} \sum_{i=1}^n W^2 \rho_i^2(t, \boldsymbol{p}) \sigma_i^2(t, \boldsymbol{p}) \right.$$
$$\left. + \sum_{i=1}^n \sum_{j=-m}^{m} (Z(t, W + \beta_{ij} W \cdot \rho_i(t, \boldsymbol{p})) - Z(t, W))\lambda_{ij}(t, \boldsymbol{p}) \right\} = -\frac{\partial Z(t, W)}{\partial t}. \tag{14}$$

Here we have used control theory for stochastic differential equations of the kind we are dealing with in this paper (see Gihman and Skorhod [16]).

If we denote the expression on the left side of (14) inside the parentheses by $L_\rho Z(t, W)$, we have under certain conditions that there exists a Markov portfolio rule $\boldsymbol{\rho}^*(t, p(t))$, $t \in [0, T]$ such that

$$\sup_{\rho \in U_{[t,T]}} L_\rho Z(t, W) = L_{\rho^*} Z(t, W). \tag{15}$$

Hence, in practice the procedure is clear:

First we solve the constrained maximization problem (15). This will leave us with a policy depending on the parameters in the price processes and on $Z(t, W)$ together with its various partial derivatives. Finally one solves the (nonlinear) partial differential equation

$$L_{\rho^*} Z(t, W) = -\frac{\partial Z(t, W)}{\partial t} \tag{16}$$

subject to the boundary condition

$$Z(T, x) = U(x). \tag{17}$$

This will give the final solution $\boldsymbol{\rho}^*$, if it exists.

Notice how we have reduced the problem of stochastic optimization to a constrained maximization problem for each t and to the solution of a partial differential equation, i.e. to problems of ordinary mathematical analyses.

Note again how matters simplify if we use the Kelly criterion: We only have to solve a constrained maximization problem for each t. The often awkward problem of solving (16) subject to (17) disappears. From (10) we have

$$\sup_{\rho} E\left\{\int_0^T \sum_{i=1}^{n} \left(\rho_i\mu_i - \tfrac{1}{2}\rho_i^2\sigma_i^2 + \sum_{j=-m}^{m} (\log(1+\rho_i\beta_{ij}))\lambda_{ij}\right) \mathrm{d}t\right\}. \tag{18}$$

Defining

$$f(\rho_1, \ldots, \rho_n)(t) = \sum_{i=1}^{n} \left(\rho_i\mu_i - \tfrac{1}{2}\rho_i^2\sigma_i^2 + \sum_{j=-m}^{m} (\log(1+\rho_i\beta_{ij}))\lambda_{ij}\right),$$

our problem is reduced to maximizing $f(\boldsymbol{\rho})(t)$ for each $t \in [0, T]$ subject to (5), say. If the Lagrangian is defined like

$$L(\rho_1, \rho_2, \ldots, \rho_n, \Lambda)(t) = f(\boldsymbol{\rho})(t) - \Lambda_t\left(\sum_{i=1}^{n} \rho_i(t) - 1\right),$$

we have to solve

$$\frac{\partial L}{\partial \rho_i} = 0, \quad i = 1, 2, \ldots, n, \quad \text{and} \quad \frac{\partial L}{\partial \Lambda} = 0.$$

The solution to this, $\boldsymbol{\rho}^*(t, \boldsymbol{p})$, is, if it exists, the desired policy (given that it satisfies some second order conditions). Hence, $\boldsymbol{\rho}^*(t, \boldsymbol{p})$ is obtained on observing the price process $\boldsymbol{p}(s)$ on $0 \leq s \leq t$ and accordingly $\boldsymbol{\rho}^*(t, \boldsymbol{p})$ is F_t-measurable.

Notice that we did not require $\boldsymbol{p}(t)$ to be Markovian. We also avoid the problem of solving the Bellman equation. The problem is reduced to a simple one of finding a constrained maximum for each t, an easy problem for a computer. The reason for this is that in (18) we got rid of $W(t)$ in the integrand due to the properties of the log-function.

What remains in the case of the Kelly criterion is the inference aspect. If we can find continuously updated estimates of the unknown parameters in the price processes, we will in practice solve the constrained maximization problem where we use estimates $\hat{\lambda}_{ij}$, $\hat{\mu}_i$ and $\hat{\sigma}_i$ instead of the unknown λ_{ij}, μ_i and σ_i's. The resulting policy $\hat{\boldsymbol{\rho}}^*(t, \boldsymbol{p})$ will thus depend, possibly directly, on the prices and indirectly on $\boldsymbol{p}(s)$, $0 \leq s \leq t$, through the estimates $\hat{\lambda}_{ij}$, $\hat{\mu}_i$, and $\hat{\sigma}_i$. Obviously, the more accurate these estimates are the closer the resulting policy will be to the optimal one.

Let us see how the theory works in the case of logarithmic utility. In this situation let us try a solution of the form

$$Z(t, W) = f(t) + \log W. \tag{19}$$

Then

$$L_{\rho}Z(t, W)=\sum_{i=1}^{n} \rho_i\mu_i-\tfrac{1}{2}\sum_{i=1}^{n} \rho_i^2\sigma_i^2+\sum_{i=1}^{n}\sum_{j=-m}^{m} (\log(1+\beta_{ij}\rho_i))\lambda_{ij}. \tag{20}$$

First we have to solve $\sup_{\rho\in U_t} L_{\rho}Z(t, W)$ and we readily see that this leads to the same ρ^* as the direct method above, since W cancels. We also get

$$f(t)=\int_0^T \sup_{\rho\in U_s} L_{\rho}Z(s)\,\mathrm{d}s-\int_0^t \sup_{\rho\in U_s} L_{\rho}Z(s)\,\mathrm{d}s \tag{21}$$

where $L_{\rho}Z(s)$ is given in (19), and this expression for $f(t)$ does not depend on W. Hence we have solved the equation (16) subject to (17) in this special case.

Another utility function of principal interest is $U(W_t)=W_t$. Here we try the separation method:

$$Z(t, W)=f(t)W. \tag{22}$$

Let us here assume our policy has to satisfy (6) in addition to (5), i.e. no borrowing or shortselling is allowed. Here

$$L_{\rho}Z(t, W)=Wf(t)\left\{\sum_{i=1}^{n} \rho_i\left(\mu_i+\sum_{k=-m}^{m} \beta_{ik}\lambda_{ik}\right)\right\}. \tag{23}$$

In solving $\sup_{\rho\in U_t} L_{\rho}Z(t, W)$, we are led to the highly speculative policy of putting everything on the single asset at time t which has the largest value of

$$\mu_i(t, \boldsymbol{p})+\sum_{k=-m}^{m} \beta_{ik}\lambda_{ik}(t, \boldsymbol{p}). \tag{24}$$

This type of solution in the case $U(W)=W$ is well known from discrete time portfolio theory (see e.g. Breiman [8]). In the time-homogeneous case, where μ_i and λ_{ik} do not depend on t, this policy might easily lead to bankruptcy from a probabilistic point of view, since $\mu_i+\sum_{k=-m}^{m} \beta_{ik}\lambda_{ik}$ might be large, but so might $\tilde{\sigma}_i$, the risk (see (11)).

Notice the similarities between the problem as it is given in (12) and the resulting Bellman equation in (14). The prices are included in the problem both directly and indirectly through the processes $\mu_i(t, \boldsymbol{p})$, $\sigma_i(t, \boldsymbol{p})$ and $\lambda_{ij}(t, \boldsymbol{p})$ since these price parameters also might depend directly on $\boldsymbol{p}$ in some measurable way. In the real time implementation on computers, we need updated estimates for these parameter processes. Hence the 'optimal' policy always depend on the latest price data information, as the case should be. On the other hand the Bellman operator does not include terms where $Z(t, W)$ is differentiated directly with respect to the p_i's. This is because the utility function only depends on W_t (in addition possibly to t if $U=U(W, t)$), but the prices are not included in the list of arguments. Hence the univariate Ito–Meyer's lemma is the one to be utilized in deriving the Bellman equation. This is not in agreement with Merton (1971, p. 629), where he implicitely assumes the utility function to be of the form $U=U(W, p_1p_2, \ldots, p_n, t)$.

4. The optimal consumption and portfolio problem

In this section we want to include consumption into the optimization problem. Here we assume that the agent wants to use some part of his fortune on consumption. Let $c(t)$ be the amount of consumption per unit time during period t. Then the stochastic differential equation we are facing is

$$\mathrm{d}W = W\sum_{i=1}^{n}\rho_i\frac{\mathrm{d}p_i}{p_i} - c(t)\,\mathrm{d}t \tag{25}$$

or

$$W_t = W_0 + \int_0^t \left[W_{s-}\sum_{i=1}^{n}\left(\rho_i\mu_i + \sum_{k=-m}^{m}\beta_{ik}\rho_i\lambda_{ik}\right) - c(s)\right]\mathrm{d}s$$
$$+\int_0^t W_{s-}\sum_{i=1}^{n}\left(\rho_i\sigma_i db_i + \sum_{k=-m}^{m}\beta_{ik}\rho_i dM_{ik}\right). \tag{26}$$

Let us pause for a moment and see where the above fits in with Harrison and Pliska's [43] comment on p. 258: Their increasing process $I_t(\phi)$ corresponds to our $\int_0^t c(s)\,\mathrm{d}s$ which has the interpretation of a consumption stream or cash flow generated by the strategy $(\boldsymbol{\rho}, c)$. An investor starting with wealth W_0 will then choose among the admissible strategies in such a way that $I(\phi)$ and W_T jointly maximize some measure of felicity: Denote by $V(c(t), t)$ the utility rate on consumption. V is assumed concave in its first argument. Let the portfolio/consumption rule be

$$\boldsymbol{u}(t, \boldsymbol{p}) = (\rho(t, \boldsymbol{p}), c(t)).$$

Then our problem is as follows:

$$\sup_{\boldsymbol{u}\in U_{[0,T]}} E\left[\int_0^T V(c(t), t)\,\mathrm{d}t + U(W(T))\right] \tag{27}$$

where $U_{[0,T]}$ is the set of permissible portfolio/consumption rules on $[0, T]$. By use of the generalized Ito's lemma and the martingale properties, this is equivalent to

$$\sup_{\boldsymbol{u}\in U_{[0,T]}} E\left[\int_0^T \left\{ V(c(t), t) + \sum_{i=1}^{n}\frac{\partial U}{\partial W}(W)(W\rho_i\mu_i) - \frac{\partial U}{\partial W}(W)c \right.\right.$$
$$+\tfrac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 U}{\partial W^2}(W)(W\rho_i\sigma_i)^2$$
$$\left.\left. + \sum_{i=1}^{n}\sum_{j=-m}^{m}[U(W(1+\rho_i\beta_{ij})) - U(W)]\lambda_{ij}\right\}\mathrm{d}t\right]. \tag{28}$$

In the present context we define the evolutional value functional

$$F_t(W, \boldsymbol{u}) = U(W(T)) + \int_t^T V(c(s), s)\,\mathrm{d}s. \tag{29}$$

Here let

$$Z(t, W) = \sup_{\boldsymbol{u}\in U_{[t,T]}} E\{F_t(W, \boldsymbol{u}) \mid W(t) = W\} \tag{30}$$

and

$$L_u Z(t, W) = \frac{\partial Z(t, W)}{\partial W}\left(\sum_{i=1}^{n} W\rho_i\mu_i - c\right) + \frac{1}{2}\frac{\partial^2 Z(t, W)}{\partial W^2}\sum_{i=1}^{n}(W\rho_i\sigma_i)^2$$

$$+ \sum_{i=1}^{n}\sum_{j=-m}^{m}(Z(t, W + W\beta_{ij}\rho_i) - Z(t, W))\lambda_{ij}. \tag{31}$$

Then we first solve the constrained maximization problem $\sup_{u \in U_{[t,T]}}[L_u Z + V]$. Under certain conditions (see Gihman and Skorohod [16, Theorem 3.21]) it follows that

$$\sup_{u \in U_{[t,T]}}\{L_u Z(t, W) + V(c, t)\} = L_{u^*} Z(t, W) + V(c^*, t). \tag{32}$$

If this is the case, one solves the associated Bellman equation

$$L_{u^*} Z(t, W) + V(c^*, t) = -\frac{\partial Z(t, W)}{\partial t} \tag{33}$$

subject to

$$Z(T, W) = U(W) \tag{34}$$

and the optimal portfolio/consumption problem has been reduced to problems in ordinary mathematical analysis.

Often the equation (33) is nonlinear and not easy to solve explicitly. As an example, consider the case with

$$U(W) = \sqrt{W}, \qquad V(c, t) = \sqrt{c(t)}$$

in the case of no jumps (i.e. the $\beta_{ij} = 0$, $i = 1, 2, \ldots, n$, $j = -m, \ldots, m$). The utility functions are strictly concave. Let us use the following notation:

$$\frac{\partial Z(t, W)}{\partial W} = Z_W, \quad \frac{\partial^2 Z(t, W)}{\partial W^2} = Z_{WW} \quad \text{and} \quad \frac{\partial Z(t, W)}{\partial t} = Z_t.$$

Let

$$f(\rho_1, \rho_2, \ldots, \rho_n, c) = Z_W W \sum_{i=1}^{n}\rho_i\mu_i - Z_W c + \tfrac{1}{2} Z_{WW} W^2 \sum_{i=1}^{n}\rho_i^2 \cdot \sigma_i^2 + \sqrt{c}$$

for each t, and

$$F(\boldsymbol{\rho}, c, \Lambda) = f(\boldsymbol{\rho}, c) - \Lambda\left(\sum_{i=1}^{n}\rho_i - 1\right)$$

where Λ is the Lagrangian multiplier. Then solving

$$\frac{\partial F}{\partial \rho_i} = 0, \quad i = 1, 2, \ldots, n, \qquad \frac{\partial F}{\partial c} = 0, \qquad \frac{\partial F}{\partial \Lambda} = 0,$$

we obtain

$$\rho_i^* = \frac{\sigma_i^{-2}}{\sum_{i=1}^n \sigma_j^{-2}} + \frac{Z_W}{Z_{WW}W} \frac{\sum_{j=1}^n \mu_j \sigma_j^{-2}}{\sigma_i^2 \sum_{j=1}^n \sigma_j^{-2}} - \frac{Z_W}{Z_{WW}W} \frac{\mu_i}{\sigma_i^2} \tag{35}$$

and

$$c^* = \frac{1}{4Z_W^2}. \tag{36}$$

Here (33) becomes

$$Z_W W \sum_{i=1}^n \rho_i^* \mu_i - \frac{1}{4Z_W} + \tfrac{1}{2} Z_{WW} \sum_{i=1}^n (W\rho_i^* \sigma_i)^2 + \frac{1}{2}\frac{1}{Z_W} = -Z_t \tag{37}$$

with

$$Z(T, W) = \sqrt{W}. \tag{38}$$

When substituting into (37) the expressions for ρ_i^*, we clearly see its nonlinear character.

In our attempt to solve (37), let us try separation of variables, i.e. assume

$$Z(t, W) = \sqrt{W} f(t) \quad \text{where } f(T) = 1.$$

Substituting this into (37), we are left with

$$-\frac{\partial f(t)}{\partial t} = \tfrac{1}{2}(f(t))^{-1} + f(t) \cdot q(t) \tag{39}$$

where

$$q(t) = \tfrac{1}{2} \sum_{i=1}^n \rho_i^*(t, \boldsymbol{p}) \mu_i(t, \boldsymbol{p}) - \tfrac{1}{8} \sum_{i=1}^n (\rho_i^*(t, \boldsymbol{p}) \sigma_i(t, \boldsymbol{p}))^2 \tag{40}$$

and

$$\rho_i^*(t, \boldsymbol{p}) = \frac{\sigma_i^{-2}}{\sum_{j=1}^n \sigma_j^{-2}} - \frac{2\sum_{j=1}^n \mu_j \sigma_j^{-2}}{\sigma_i^2 \sum_{j=1}^n \sigma_j^{-2}} + \frac{2\mu_i}{\sigma_i^2}. \tag{41}$$

Note that (41) is in agreement with the discrete time result stating that for iso-elastic marginal utility the portfolio selection decision does not depend on the consumption decision (Samuelson [34]).

Since $q(t)$ does not depend on W, the separation method worked successfully in this case. Notice that we have already completely determined the optimal portfolio rule in (41). In order to find the optimal consumption $c^*(t)$, we only have to solve (39) subject to $f(T) = 1$, and insert into (36), noticing that $Z_W = f(t)/2\sqrt{W}$. The nonlinear differential equation (39) is of Bernoulli type. Using the substitution

$g(t)=(f(t))^2$, (39) reduces to

$$\frac{\partial g(t)}{\partial t}-2q(t)g(t)=1$$

which is an ordinary, linear nonhomogeneous differential equation that can be solved by quadrature. Here we find

$$f(t)=\left(\exp\left\{-\int_t^T 2q(s)\,\mathrm{d}s\right\}+\int_t^T \exp\left\{-\int_t^s 2q(v)\,\mathrm{d}v\right\}\mathrm{d}s\right)^{1/2}. \tag{42}$$

Hence our optimal consumption rule is also determined explicitly as

$$c^*(t)=\frac{W_t}{\mathrm{e}^{-\int_t^T 2q(s)\mathrm{d}s}+\int_t^T \mathrm{e}^{-\int_t^s 2q(v)\mathrm{d}v}\,\mathrm{d}s}. \tag{43}$$

From (43) we see that marginal propensity to consume equals a constant (time-dependent) proportion of wealth.

The problem of including noncapital gains income (wage) in the basic equation (25), is easily solved in the present framework. Let $\mathrm{d}g(t)=h(t)\,\mathrm{d}t$ equal the instantaneous flow of noncapital gains income. Then our new basic equation becomes

$$\mathrm{d}W(t)=W(t)\sum_{i=1}^{n}\rho_i\frac{\mathrm{d}p_i}{p_i}-(c(t)-h(t))\,\mathrm{d}t. \tag{44}$$

Here the rate $h(t)$ does not depend on $W(t)$ and is assumed known at each t. Hence our new problem is translated back to our previous one by using $c'=c-h$ and $V'(c',t)=V(c,t)+h(t)$. In principle we face the same type of mathematical optimization problem as before, but the computational difficulties might change. For example is the equation (37) no longer solvable by separating the variables W and t as demonstrated above.

In the usual gambling situation it is well known that one should keep a reserve fund while gambling in order to avoid ruin. The theoretical problem is then that one is no longer following any particular 'optimal strategy'. In the present setting this could be resolved by considering the reserve fund as part of the optimal consumption $c^*(t)$. One other possibility is to let the reserve fund compete on the market with the other investment alternatives, then with $\tilde{\sigma}_{rf}=0$, where $\tilde{\sigma}_{rf}$ represents the risk of the reserve fund. One can also combine these two possibilities. Thus way the reserve fund will vary over time, and presumably in an optimal fashion.

If the n-th asset represents the reserve fund and is 'risk free', we assume $\sigma_n=0$ and $\beta_{nk}=0$, $k=-m,-m+1,\ldots,m$. (See (11).) Here our basic stochastic differential equation becomes

$$\begin{aligned}\mathrm{d}W_t=&\sum_{i=1}^{n-1}W_t\left\{\rho_i(\mu_i-\mu_n)+\rho_i\sum_{k=-m}^{m}\beta_{ik}\lambda_{ik}\right\}\mathrm{d}t\\&+\sum_{i=1}^{n-1}W_t\left\{\rho_i\sigma_i\,\mathrm{d}b_i+\rho_i\sum_{k=-m}^{m}\beta_{ik}\,\mathrm{d}M_{ik}\right\}+\{\mu_nW_t+h_t-c_t\}\,\mathrm{d}t.\end{aligned} \tag{45}$$

Now the relation $\rho_n = 1 - \sum_{i=1}^{n-1} \rho_i$ will ensure that the identity constraint $\sum_{i=1}^{n} \rho_i = 1$ is satisfied, no matter what values $\rho_1, \ldots, \rho_{n-1}$ take; i.e. if we allow borrowing and shortselling (see Merton [26]). Hence the problem with a reserve fund can be viewed as an unconstrained maximization problem, changing the set $U_{[0,T]}$. (On the other hand, if we impose (6), i.e. $\rho_i \geq 0$, $i = 1, 2, \ldots, n$, then the problem is still a constrained one.)

To illustrate, let us again consider the example with $U(W) = \sqrt{W}$, $V(c, t) = \sqrt{c}$; no jumps in the p_i's and no wages. The unconstrained maximization now gives us

$$\rho_i^* = -\frac{Z_W}{WZ_{WW}} \frac{\mu_i}{\sigma_i^2}, \quad i = 1, 2, \ldots, n-1, \qquad c^* = \frac{1}{4Z_W^2}.$$

The partial differential equation is

$$Z_W \left(W \sum_{i=1}^{n-1} \rho_i^*(\mu_i - \mu_n) - c^* + \mu_n W \right) + \tfrac{1}{2} Z_{WW} W^2 \sum_{i=1}^{n-1} (\rho_i^* \sigma_i)^2 + \sqrt{c^*} = -Z_t,$$

and trying $Z(t, W) = \sqrt{W} f(t)$, we are again successful in separating the variables and obtain for the optimal portfolio and consumption rates

$$\rho_i^* = 2\sigma_i^{-2}(\mu_i - \mu_n), \quad i = 1, 2, \ldots, n-1, \qquad \rho_n^* = 1 - \sum_{j=1}^{n} \rho_j^* \tag{46}$$

and

$$c^*(t) = W_t \cdot \left\{ \exp\left[-\int_t^T \left(\sum_{i=1}^{n-1} \sigma_i^{-2}(\mu_i - \mu_n)^2 + \mu_n \right) \mathrm{d}s \right] + \int_t^T \exp\left[-\int_t^s \left(\sum_{i=1}^{n-1} \sigma_i^{-2}(\mu_i - \mu_n)^2 + \mu_n \right) \mathrm{d}u \right] \mathrm{d}s \right\}^{-1}. \tag{47}$$

Notice how the risk σ_i on asset i affects the optimal portfolio ratio ρ_i^*; the higher the risk on asset i, the less absolute value of the portfolio ratio no. i.

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References

[1] O.O. Aalen, Statistical Inference for a Family of Counting Processes (Inst. Math. Statist., Univ. Copenhagen, 1976).

[2] K.K. Aase, Model references adaptive systems applied to regression analyses, Statistica Neerlandica 3 (1981) 129–155.
[3] K.K. Aase, Stochastic continuous time model reference adaptive systems with decreasing gain, Adv. Appl. Prob. 14 (1982) 763–788.
[4] K.K. Aase, Recursive estimation in nonlinear time series models of autoregressive type, Journal of the Royal Statist. Soc., Ser. B 45(2) (1983) 228–237.
[5] Bharucha-Reid, ed., Probabilistic Analyses and related topics (Academic Press, New York, 1979).
[6] P. Billingsley, Convergence of Probability Measures (Wiley, New York, 1968).
[7] K. Borch, The Economics of Uncertainty (Princeton University Press, Princeton, NJ, 1968).
[8] L. Breiman, Investment policies for expanding businesses optimal in a long run sense, Naval Research Logistics Quarterly 7(4) (1960) 647–651.
[9] L. Breiman, Optimal Gambling Systems for Favorable Games, in: 4th Berkeley Symposium on Probability and Statistics 1 (1961), 65–78.
[10] D.R. Brillinger, Comparative aspects of the study of ordinary time series and of point processes, in: Developments in statistics, vol. 1 (Department of Statistics, University of California at Berkeley, Berkeley, CA, 1978).
[11] B.M. Brown and J.I. Hewitt, Asymptotic likelihood theory for diffusion processes. J. Appl. Prob. 12 (1975) 228–238.
[12] A.S. Dexter, J.N.W. Yu and W.T. Ziemba, Portfolio selection in a lognormal market when the investor has a power utility function: Computational results, in: M.A.H. Dempster, ed., Stochastic Programming (Academic Press, London, 1980).
[13] J.L. Doob, Stochastic Processes. (Wiley, New York, 1953).
[14] E.B. Dynkin and A.A. Yushkevich, Controlled Markov Processes (Springer-Verlag, Berlin, Heidelberg, New York, 1979).
[15] I.I. Gihman and A.V. Skorohod, Densities of probability measures in function spaces, Russian Math. Surveys 21 (6) (1966) 83–156.
[16] I.I. Gihman and A.V. Skorohod, Controlled Stochastic Processes (Springer-Verlag, Berlin, Heidelberg, New York, 1979).
[17] I.I. Gihman and A.V. Skorohod, The Theory of Stochastic Processes vol. III (Springer-Verlag, Berlin, Heidelberg, New York, 1979).
[18] N.H. Hakanson, Optimal investment and consumption strategies under risk for a class of utility functions, Econometrica 38 (5) (1970) 585–607.
[19] J. Jacod and J. Memin, Characteristiques Locales et Conditions de Continuity Absolute Pour les Semi-Martingales (Dep. Math. Inform. Univ. de Rennes, 1975).
[20] Yu,M. Kabanov, R.S. Lipster and A.N. Shiryayev, Criteria for absolute continuity of measures of multivaraite point processes III, (Lecture notes Springer-Verlag, Berlin, 1976).
[21] N. Keiding, Maximum likelihood estimation in the birth-and-death process, Ann. Statist. 3 (1975) 363–372.
[22] T.L. Lindstrøm, Hyperfine stochastic integration: The Nonstandard Theory, Preprint no. 2 (Department of mathematics, University of Oslo, Norway, 1979).
[23] H.P. McKean, Jr., Stochastic Integrals (Academic Press, New York, 1969).
[24] H.M. Markowitz, Portfolio Selection. Efficient Diversification of Investments. 4th printing 1976 (New Haven and London, Yale University Press, 1958).
[25] R.C. Merton, Lifetime portfolio selection under uncertainty: The continuous-time case. Rev. Economics and Statistics (1969) 247–257.
[26] R.C. Merton, Optimum consumption and portfolio rules in a continuous-time model, Journ. of Economic Theory 3 (4) (1971) 373–413.
[27] P.A. Meyer, Un Cours sur les Intégrales Stochastiques, in: P.A. Meyer, ed., Seminaire de Probabilité X, LNM 511 (Springer-Verlag, Berlin, 1976) 245–400.
[28] J. Mossin, Optimal multiperiod portfolio polices, Journal of Business 41 (2) (1968) 215–229.
[29] M.F.M. Osborne, Brownian motion in the stock market, Operations Research March–April 1959) 145-173.
[30] J.W. Pratt, Risk aversion in the small and in the large. Econometrica 32 (1–2) (1964) 122–127.
[31] M.M. Rao, Stochastic processes and integration. (Sijthoff and Nordhoff, Alphen aan de Rijn, 1979).
[32] P.A. Samuelson, Risk under uncertainty: A fallacy of large numbers, Scientica, 6th series, 57th year (1963).

[33] P.A. Samuelson, General proof that diversification pays, Journ. Financial and Quant. Analyses II (1967) 1–13.
[34] P.A. Samuelson, Lifetime portfolio selection by stochastic programming, Rev. Economics and Statistics (1969) 239–246.
[35] P.A. Samuelson, The fundamental approximation theorem of portfolio analyses in terms of means, variances and higher moments, Review of Economic Studies (1970) 537–542.
[36] A.F. Taraskin, On the asymptotic normality of vector-valued stochastic integrals and estimates of drift parameters of a multi-dimensional diffusion process, Theor. Probability and Math. Statist. 2 (1974) 209–224.
[37] E.O. Thorp, Portfolio Choice and the Kelly Criterion (1971), in: W.T. Ziemba and R.G. Vickson, eds., Stochastic Optimization Models in Finance (Academic Press, New York, 1975) 599–619.
[38] G. Tintner and J.K. Senegupta, Stochastic Economics (Academic Press, New York, 1972).
[39] J. Tobin, The theory of portfolio selection, in: F.H. Hahn and F.P.R. Brechling, eds., The theory of interest rates (Macmillan, London, 1965).
[40] J. Tobin, Liquidity preference as behaviour towards risk. Review of Economic Studies (1958) 65–86.
[41] P. Varaiya, The martingale theory of jump processes. IEEE Trans. Autom. Contr. 20 (1) (1975) 34–42.
[42] E. Wong, Recent progress in stochastic processes – a survey, IEEE Trans. Inform. Theory 19 (3) (1973) 262–275.
[43] J.M. Harrison and S.R. Pliska, Martingales and stochastic integrals in the theory of continuous trading. Stochastic Processes and their Applications 11 (1981) 215–260.